

# GEOMETRIC CONFIGURATION OF RIEMANNIAN SUBMANIFOLDS OF ARBITRARY CODIMENSION

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**ABSTRACT.** In this paper we study a geometric configuration of submanifolds of arbitrary codimension in an ambient Riemannian space. We obtain relations between the geometry of a  $q$ -codimension submanifold  $M^n$  along its boundary and the geometry of the boundary  $\Sigma^{n-1}$  of  $M^n$  as an hypersurface of a  $q$ -codimensional submanifold  $P^n$  in an ambient space  $\overline{M}^{n+q}$ . As a consequence of these geometric relations we get that the ellipticity of the generalized Newton transformations implies the transversality of  $M^n$  and  $P^n$  in  $P^n$  is totally geodesic in  $\overline{M}^{n+q}$ .

## 1. INTRODUCTION

Let  $\overline{M}^{n+q}$  be  $n + q$ -dimensional connected and orientable Riemannian manifold with metric  $\langle \cdot, \cdot \rangle$  and Levi-Civita connection  $\overline{\nabla}$ . Denote by  $P^n$  an oriented connected  $n$ - submanifold of  $\overline{M}^{n+q}$  and consider  $\Sigma^{n-1}$  an  $n - 1$ -compact hypersurface of  $P^n$ . If  $\Psi : M^n \rightarrow \overline{M}^{n+q}$  is an oriented connected and compact submanifold of  $\overline{M}^{n+q}$  with boundary  $\partial M$ .  $M^n$  will be said submanifold of  $\overline{M}^{n+q}$  with boundary  $\Sigma^{n-1}$  if the restriction of  $\Psi$  to  $\partial M$  is a diffeomorphism onto  $\Sigma^{n-1}$ . A natural question would be: How can one describe the geometry of  $M^n$  along its boundary  $\partial M$  with respect to the geometry of the inclusion  $P^n \subset \overline{M}^{n+q}$ ? A partial answer to this question is given by the following formula, obtained in this paper, which holds along the boundary  $\partial M$ : for any multi-index  $u = (u_1, \dots, u_q)$  with length  $|u| \leq n - 1$ ,

$$(1.1) \quad \langle T_u \nu, \nu \rangle = \frac{1}{n - 1 - |u|} \sum_{l \leq u} \binom{n - 1 - |l|}{|u| - l} \rho^l \mu^{u-l} \sigma_{|l|}(A_\Sigma).$$

where  $(T_u)_u$  stands for the family of the generalized Newton transformations introduced in [4] associated to the matrix  $A = (A_1, \dots, A_q)$ ;  $(A_\alpha)_{\alpha \in \{1, \dots, q\}}$  is a system of matrices of the shape operators corresponding to a normal basis to the manifold  $M^n$  and  $\tilde{\sigma}_u = \tilde{\sigma}_u(A_1|_\Sigma, \dots, A_q|_\Sigma)$  are the coefficients of the

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Newton polynomial  $P_{\tilde{A}} : \mathbb{R}^q \longrightarrow \mathbb{R}$  defined by

$$P_{\tilde{A}}(t) = \sum_{|u| \leq n-1} \tilde{\sigma}_u t^u$$

where  $\tilde{A} = (A_1|_{\Sigma}, \dots, A_q|_{\Sigma})$ ,  $A_{\alpha}|_{\Sigma}$  is the restriction of  $A_{\alpha}$  to  $\Sigma^{n-1}$  and  $\sigma_r(A_{\Sigma})$  is the symmetric function coefficient of  $t^r$  in the characteristic polynomial of the matrix  $A_{\Sigma}$ .

In [2] Alías and Malacarne considered the above geometric configuration to the case of hypersurfaces where  $\Sigma_{n-1}$  is an  $(n-1)$ -dimensional compact submanifold contained in an hyperplane  $\Pi$  of  $\mathbb{R}^{n+1}$  and  $M^n$  stands for a smooth compact, connected and oriented manifold with boundary  $\partial M^n$ . Moreover  $M^n$  is an hypersurface of  $\mathbb{R}^{n+1}$  with boundary  $\Sigma_{n-1}$  in the sense that there exists  $\psi : M^n \rightarrow \mathbb{R}^{n+1}$  an oriented hypersurface immersed in  $\mathbb{R}^{n+1}$  such that the restriction of  $\psi$  to the boundary  $\partial M^n$  is a diffeomorphism onto  $\Sigma_{n-1}$ . They showed that along the boundary  $\partial M^n$ , for every  $1 \leq r \leq n-1$ :

$$(1.2) \quad \langle T_r \nu, \nu \rangle = (-1)^r s_r \langle a, \nu \rangle^r$$

where  $\nu$  stands for the outward pointing unit conormal vector field along  $\partial M^n$  while  $T_r$  denotes the classical Newton transformation,  $a \in \mathbb{R}^{n+1}$  such that  $\Pi = a^{\perp}$  and  $s_r$  is the  $r$ -th symmetric function of the principal curvatures of  $\Sigma_{n-1}$  with respect to  $\nu$ . In [3] Alías, de Lira and Malacarne studied the question in the context of an  $(n+1)$ -dimensional connected oriented ambient Riemannian manifold  $\overline{M}^{n+1}$ , they established that along the boundary  $\partial M^n$ , for every  $1 \leq r \leq n-1$ :

$$(1.3) \quad \langle T_r \nu, \nu \rangle = (-1)^r s_r \langle \xi, \nu \rangle^r$$

where  $T_r$ ,  $\nu$ ,  $s_r$ , are as in relation (1.2) where  $P^n \subset \overline{M}^{n+1}$  is an embedded totally geodesic submanifold instead of the hyperplane  $\Pi$  and  $\xi$  is a unitary normal vector field to  $P^n$ . Relation (1.3) shows that the ellipticity of the Newton transformation  $T_r$ , for some  $1 \leq r \leq n-1$  on  $M^n$ , implies the transversality of the hypersurfaces  $M^n$  and  $P^n$  along their boundary. Formula (1.3) was also obtained, in [1], by the two first authors in context of pseudo-Riemannian spaces. Moreover we deduce from relation (1.1) that the ellipticity of the generalized Newton transformation  $T_u$  implies the transversality of the  $q$ -codimension submanifolds  $M^n$  and  $P^n$  in case where  $P^n$  is totally geodesic submanifold of  $\overline{M}^{n+q}$ .

## 2. PRELIMINARIES

In this section, we will recall some properties of the generalized Newton transformations and we will show how our method works.

**2.1. Generalized Newton Transformations.** Let  $E$  be an  $n$ -dimensional real vector space and  $End(E)$  be the vector space of endomorphisms of  $E$ . Denote by  $\mathbb{N}$  the set of nonnegative integers and let  $\mathbb{N}^q$  be the one of multi-index  $u = (u_1, \dots, u_q)$  with  $u_j \in \mathbb{N}$ . The length  $|u|$  of  $u$  is given by

$|u| = u_1 + \dots + u_q$ .  $End^q(E)$  stands for the vector space  $End(E) \times \dots \times End(E)$   $q$ -times. For  $A = (A_1, \dots, A_q) \in End^q(E)$ ,  $t = (t_1, \dots, t_q) \in R^q$  and  $u \in \mathbb{N}^q$ , we set

$$\begin{aligned} tA &= t_1 A_1 + \dots + t_q A_q \\ t^u &= t_1^{u_1} \dots t_q^{u_q}. \end{aligned}$$

For  $\alpha \in \{1, \dots, q\}$  we define ( see [4] ) the musical functions  $\alpha_b : \mathbb{N}^q \longrightarrow \mathbb{N}^q$  and  $\alpha^\sharp : \mathbb{N}^q \longrightarrow \mathbb{N}^q$  by

$$\alpha_b(i_1, \dots, i_q) = (i_1, \dots, i_{\alpha-1}, i_\alpha - 1, i_{\alpha+1}, \dots, i_q)$$

and

$$\alpha^\sharp(i_1, \dots, i_q) = (i_1, \dots, i_{\alpha-1}, i_\alpha + 1, i_{\alpha+1}, \dots, i_q)$$

It is clear that  $\alpha_b$  is the inverse map of  $\alpha^\sharp$ .

The generalized Newton transformation (GNT in brief ) is a system of endomorphisms  $T_u = T_u(A)$ ,  $u \in \mathbb{N}^q$ , that satisfies the following recursive relations

$$\begin{aligned} T_0 &= I & \text{where } 0 &= (0, \dots, 0), \\ T_u &= \sigma_u I - \sum_{\alpha} A_{\alpha} T_{\alpha_b(u)} & \text{where } |u| > 1 \\ &= \sigma_u I - \sum_{\alpha} T_{\alpha_b(u)} A_{\alpha} \end{aligned}$$

where  $\sigma_u$  are the coefficients of the Newton polynomial  $P_A : \mathbb{R}^q \longrightarrow \mathbb{R}$  of  $A$ , given by

$$P_A(t) = \det(I + tA) = \sum_{|u| \leq n} \sigma_u t^u$$

$\sigma_u = \sigma_u(A_1, \dots, A_q)$  depends only on  $A = (A_1, \dots, A_q)$  and  $I$  is the identity map on  $E$ .

**2.2. The method.** We will describe how our method works.

**2.2.1. Hypersurfaces' case.** Let  $M^n$  be a  $n$ -submanifold of codimension one in  $\overline{M}^{n+1}$  of boundary  $\partial M$ . Assume the boundary  $\Sigma^{n-1} = \partial M$  is a codimension one in  $P^n \subset \overline{M}^{n+1}$ . Then we have the inclusions

$$\Sigma^{n-1} \subset M^n \subset \overline{M}^{n+1}, \Sigma^{n-1} \subset P^n \subset \overline{M}^{n+1}.$$

Denote the corresponding shape operators, respectively, by

$$A_{\Sigma}, A_P, A_{\Sigma, P}, A.$$

In our consideration we will need only  $A_{\Sigma}$ ,  $A_P$ ,  $A$ . More precisely we will use

$$A_{\Sigma}, A_P|_{\Sigma}, A.$$

First two are represented by square matrices of dimension  $n - 1$  whereas the last one by a square matrix of dimension  $n$ . The intrinsic geometry of  $\Sigma^{n-1}$

in  $M^n$  is coded in the pair  $(A_\Sigma, A_P|_\Sigma)$  and the geometry of  $M^n \subset \overline{M}^{n+1}$  is given by  $A$ . Therefore we will use the following Newton Transformation and the generalized Newton Transformations

$$T_{(k,l)} = T_{(k,l)}(A_\Sigma, A_P|_\Sigma) \text{ and } T_r = T_r(A)$$

and corresponding symmetric functions

$$\sigma_{(k,l)} = \sigma_{(k,l)}(A_\Sigma, A_P|_\Sigma) \text{ and } \sigma_r = \sigma_r(A).$$

The goal is to show that

$$(2.1) \quad \langle T_r \nu, \nu \rangle = \sum_{k+l=r} \sigma_{(k,l)}$$

where  $\nu$  is the unit normal vector to  $\Sigma^{n-1}$  in  $M^n$ .

The only geometric considerations involved are the ones which lead to the formulas

$$\langle N, \nu \rangle = \langle \xi, N \rangle, \quad \langle \eta, N \rangle = -\langle \xi, \nu \rangle$$

and

$$\langle Ae_i, e_j \rangle = -\langle A_\Sigma e_i, e_j \rangle \langle \xi, \nu \rangle + \langle A_P e_i, e_j \rangle \langle \xi, N \rangle$$

where  $N$  is unit normal vector with the respect to inclusion  $M^n \subset \overline{M}^{n+1}$ ,  $\xi$  unit normal vector with respect to  $P^n \subset \overline{M}^{n+1}$  and  $\eta$  is the unit normal vector of  $\Sigma^{n-1} \subset P^n$  and  $(e_1, \dots, e_{n-1})$  is a local orthonormal basis of  $T\Sigma^{n-1}$ , we may assume that this basis consists of eigenvectors of  $A_\Sigma$  i.e.  $A_\Sigma e_i = \tau_i e_i$ . In other words

$$A|_\Sigma = -\langle \xi, \nu \rangle A_\Sigma + \langle \xi, N \rangle A_P|_\Sigma.$$

Assuming  $P$  is totally umbilical in  $\overline{M}^{n+1}$ , we have  $A_P|_\Sigma = \lambda I_{T\Sigma^{n-1}}$ . Hence

$$(2.2) \quad A|_\Sigma = -\langle \xi, \nu \rangle A_\Sigma + \lambda \langle \xi, N \rangle I_{T\Sigma^{n-1}}.$$

Denote by  $\tilde{A}$  the matrix of  $A$  with the respect of the basis  $(e_1, \dots, e_{n-1}, \nu)$  and by  $A$  the matrix of  $A|_\Sigma$ . Then

$$\tilde{A} = \begin{pmatrix} A & B \\ B^\top & c \end{pmatrix}, \text{ where } B = \begin{pmatrix} \langle A\nu, e_1 \rangle \\ \vdots \\ \langle A\nu, e_{n-1} \rangle \end{pmatrix} \text{ and } c = \langle A\nu, \nu \rangle.$$

Let us compare symmetric functions of  $\tilde{A}$  with symmetric functions of  $A$ . We have

$$\begin{aligned} P_{\tilde{A}}(t) &= \det \begin{pmatrix} I_{n-1} + tA & tB \\ tB^\top & 1 + tc \end{pmatrix} = (1 + tc - t^2 B^\top (I_{n-1} + tA)^{-1} B) \det(I_{n-1} + tA) \\ &= f(t) P_A(t), \end{aligned}$$

where  $f(t) = 1 + tc - t^2 B^\top (I_{n-1} + tA)^{-1} B$ . Recall that

$$P_{\tilde{A}}(t) = \sum_{j=0}^n (-1)^j \sigma_j(\tilde{A}) t^j.$$

Hence

$$\begin{aligned} (-1)^r r! \sigma_r(\tilde{A}) &= \frac{d^r}{dt^r} P_{\tilde{A}}(0) = \sum_{j=0}^r \binom{r}{j} \frac{d^{r-j}}{dt^{r-j}} P_A^{(r-j)}(0) f^{(j)}(0) \\ &= \sum_{j=0}^r \binom{r}{j} (-1)^{r-j} (r-j)! \sigma_{r-j}(A) f^{(j)}(0). \end{aligned}$$

It is not hard to see that

$$f(0) = 1, f'(0) = -c \text{ and } f^{(j)}(0) = -j! B^\top A^{j-2} B \text{ for } j \geq 2.$$

Therefore

$$(2.3) \quad \sigma_r(\tilde{A}) = \sigma_r(A) + c \sigma_{r-1}(A) - \sum_{j=2}^r (-1)^j (B^\top A^{j-2} B) \sigma_{r-j}(A).$$

Let us now move to symmetric functions of two matrices. Notice first that

$$\sigma_r(\tilde{A} + \lambda I_n) = \sum_{j=0}^r \binom{n-j}{r-j} \lambda^{r-j} \sigma_j(\tilde{A}).$$

Indeed,  $P_{\tilde{A} + \lambda I_n}(t) = (1 + t(a_1 + \lambda)) \dots (1 + t(a_n + \lambda))$  if  $a_1, \dots, a_n$  are the eigenvalues of  $\tilde{A}$ . Notice moreover that

$$\begin{aligned} P_{\tilde{A} + \lambda I_n}(t) &= \det(I_n + t\tilde{A} + s\lambda I_n) \\ &= (1 + ta_1 + s\lambda) \dots (1 + ta_n + s\lambda). \end{aligned}$$

Thus

$$(2.4) \quad \sigma_{(k,l)}(\tilde{A}, \lambda I_n) = \binom{n-k}{l} \lambda^l \sigma_k(\tilde{A}).$$

Hence

$$(2.5) \quad \sigma_r(\tilde{A} + \lambda I_n) = \sum_{j=0}^r \sigma_{(j,r-j)}(\tilde{A}, \lambda I_n)$$

We need to show, by (2.5) and (2.1), that

$$(2.6) \quad \langle T_r \nu, \nu \rangle = \tilde{\sigma}_r(A).$$

By the recurrence formula for  $T_r$  we have

$$\begin{aligned} \langle T_r \nu, \nu \rangle &= \sigma_r(\tilde{A}) - \langle T_{r-1} \nu, \tilde{A} \nu \rangle \\ &= \sigma_r(\tilde{A}) - c \langle T_{r-1} \nu, \nu \rangle - \sum_{i=1}^{n-1} \langle T_{r-1} \nu, e_i \rangle \langle \tilde{A} \nu, e_i \rangle. \end{aligned}$$

We will show that

$$(2.7) \quad \sum_{i=1}^{n-1} \langle T_k \nu, e_i \rangle \langle \tilde{A} \nu, e_i \rangle = - \sum_{j=2}^k (-1)^j (B^\top A^{j-2} B) \sigma_{k-j}(A).$$

Indeed, assuming  $(e_i)$  is a basis consisting of eigenvectors with eigenvalues  $(\tau_i)_i$  by induction we have

$$\begin{aligned} \sum_{i=1}^{n-1} \langle T_k \nu, e_i \rangle \langle \tilde{A} \nu, e_i \rangle &= -b_i^2 \sum_{j=2}^k (-j) \sigma_{k-j}(A) \tau_i^{j-2} \\ &= \sum_{j=2}^k (-1)^j \left( b_i \tau_i^{j-2} b_i \right) \sigma_{k-j}(A) \end{aligned}$$

where  $B^\top = (b_i)_i$ , which proves (2.7). Thus by induction and (2.3), we get

$$\langle T_r \nu, \nu \rangle = \sigma_r(\tilde{A}) - c \sigma_{r-1}(A) + \sum_{j=2}^r (-1)^j (B^\top A^{j-2} B) \sigma_{r-j}(A) = \tilde{\sigma}_r(A).$$

**2.2.2. General case.** We generalize considerations to any codimension  $q$ . We generalize equations (2.1) and (2.6). Let  $P^n$  and  $M^n$  be of codimension  $q$  in  $\overline{M}^{n+q}$ . We assume that  $P^n$  is totally umbilical in  $\overline{M}^{n+1}$ . As before  $\Sigma^{n-1} \subset P^n$  is a boundary of  $M^n$ . Then we have the following shape operators

$$A_\Sigma, A_P^{\xi_1}, \dots, A_P^{\xi_p}, A^{N_1}, \dots, A^{N_q},$$

corresponding to inclusions  $\Sigma^{n-1} \subset P^n$ ,  $P^n \subset \overline{M}^{n+q}$ , and  $M^n \subset \overline{M}^{n+q}$ , where  $(\xi_1, \dots, \xi_q)$  are orthogonal to  $P^n$  and  $(N_1, \dots, N_q)$  are orthogonal to  $M^n$ . Let

$$T_u = T_u(A^{N_1}, \dots, A^{N_q}), \quad T_v = T_v(A_\Sigma, A^{\xi_1}|_\Sigma, \dots, A^{\xi_q}|_\Sigma)$$

and

$$\widetilde{T}_u = \widetilde{T}_u(A^{N_1}|_\Sigma, \dots, A^{N_q}|_\Sigma)$$

where  $u$  is of length  $q$  and  $v$  of length  $q+1$ . We hope that the following relations hold

$$(2.8) \quad \langle T_u \nu; \nu \rangle = \tilde{\sigma}_u$$

and

$$\tilde{\sigma}_u = \sum_{|v|=|u|, v \text{ not increasing}} c_v \sigma_v$$

where  $c_v$  is a coefficient independent of  $(A^{N_1}|_\Sigma, \dots, A^{N_q}|_\Sigma)$ . Both of them imply

$$\langle T_u \nu; \nu \rangle = \sum_{|v|=|u|, v \text{ not increasing}} c_v \sigma_v.$$

The first equality is a generalization of (2.6), whereas the second one of (2.5). Moreover, the first one is purely algebraic and the second one uses correspondence between  $A_\Sigma, A^{\xi_1}|_\Sigma, \dots, A^{\xi_q}|_\Sigma$  and  $A^{N_1}|_\Sigma, \dots, A^{N_q}|_\Sigma$  analogous to the relation (2.2) in the codimension one case.

Let us be more precise. First notice that

$$\begin{aligned}
\bar{\nabla}_{e_i} e_j &= \sum_{k=1}^{n-1} \langle \bar{\nabla}_{e_i} e_j, e_k \rangle e_k + \langle \bar{\nabla}_{e_i} e_j, \nu \rangle \nu + \sum_{\alpha=1}^q \langle \bar{\nabla}_{e_i} e_j, N_\alpha \rangle N_\alpha \\
&= \sum_{k=1}^{n-1} \langle \bar{\nabla}_{e_i} e_j, e_k \rangle e_k + \langle \bar{\nabla}_{e_i} e_j, \nu \rangle \nu + \sum_{\alpha=1}^q \langle A^{N_\alpha} e_i, e_j \rangle N_\alpha
\end{aligned}$$

and

$$\begin{aligned}
\bar{\nabla}_{e_i} e_j &= \sum_{k=1}^{n-1} \langle \bar{\nabla}_{e_i} e_j, e_k \rangle e_k + \langle \bar{\nabla}_{e_i} e_j, \eta \rangle \eta + \sum_{\alpha=1}^q \langle \bar{\nabla}_{e_i} e_j, \xi_\alpha \rangle \xi_\alpha \\
&= \sum_{k=1}^{n-1} \langle \bar{\nabla}_{e_i} e_j, e_k \rangle e_k + \langle A_\Sigma e_i, e_j \rangle \eta + \sum_{\alpha=1}^q \langle A^{\xi_\alpha} e_i, e_j \rangle \xi_\alpha
\end{aligned}$$

Thus

$$\langle \bar{\nabla}_{e_i} e_j, \nu \rangle \nu + \sum_{\alpha=1}^q \langle A^{N_\alpha} e_i, e_j \rangle N_\alpha = \langle A_\Sigma e_i, e_j \rangle \eta + \sum_{\alpha=1}^q \langle A^{\xi_\alpha} e_i, e_j \rangle \xi_\alpha$$

Hence

$$\begin{aligned}
\langle A^{N_\alpha} e_i, e_j \rangle &= \langle \eta, N_\alpha \rangle \langle A_\Sigma(e_i), e_j \rangle + \sum_{\beta=1}^q \langle \xi_\beta, N_\alpha \rangle \langle (A^{\xi_\beta}) e_i, e_j \rangle \\
&= \langle \eta, N_\alpha \rangle \langle A_\Sigma(e_i), e_j \rangle + \sum_{\beta=1}^q \langle \xi_\beta, N_\alpha \rangle \langle A^{\xi_\beta} e_i, e_j \rangle
\end{aligned}$$

Assuming that  $P^n$  is totally umbilical, i.e.  $A^{\xi_\alpha} = \lambda_\alpha I_n$ , where  $I_n$  denotes the identity map of the tangent space  $T_p P^n$ , we get

$$A^{N_\alpha}|_\Sigma = \langle \eta, N_\alpha \rangle A_\Sigma + \sum_{\beta=1}^q \langle \xi_\beta, N_\alpha \rangle \lambda_\beta I_{n-1}.$$

Notice that it can be written in the form

$$(2.9) \quad A^{N_\alpha}|_\Sigma = \langle \eta, N_\alpha \rangle A_\Sigma + \langle V, N_\alpha \rangle I_{n-1}, \quad \text{where } V = \sum_{\beta=1}^q \lambda_\beta \xi_\beta.$$

Moreover one can show that

$$\langle \eta, V \rangle = \det (\langle \xi_\alpha, N_\beta \rangle)_{\alpha, \beta} \quad \text{and} \quad \langle \eta, N_\alpha \rangle = -\det C_\alpha$$

where  $C_\alpha$  is a matrix obtained from  $C = (\langle \xi_\alpha, N_\beta \rangle)_{\alpha, \beta}$  by replacing the  $\alpha$ -th column by  $\langle \xi_\alpha, \nu \rangle$ .

Hence we have

$$\tilde{T}_u = \tilde{T}_u (\rho_1 A_\Sigma + \mu_1 I_{n-1}, \dots, \rho_q A_\Sigma + \mu_q I_{n-1}), \quad \text{where } \rho_\alpha = \langle \eta, N_\alpha \rangle, \mu_\alpha = \langle V, N_\alpha \rangle.$$

Now if  $\{e_1, \dots, e_{n-1}\}$  is a basis of eigenvectors of the shape operator  $A_\Sigma$  then, for every  $i \in \{1, \dots, n-1\}$ ,

$$A_\Sigma(e_i) = \tau_i e_i.$$

where  $\tau_i$  are the corresponding eigenvalues.

By relation (2.9) we get

$$\begin{aligned} \langle A_\alpha(e_i), e_j \rangle &= (\langle \eta, N_\alpha \rangle \tau_i + \langle V, N_\alpha \rangle) \delta_i^j \\ &= \gamma_{i,\alpha} \delta_i^j, \end{aligned}$$

where

$$\gamma_{i,\alpha} = \langle \eta, N_\alpha \rangle \tau_i + \langle V, N_\alpha \rangle.$$

Thus the matrix associated to the shape operator  $A_\alpha$  with respect to the basis  $\{e_1, \dots, e_{n-1}, \nu\}$  is given by

$$A_\alpha = \begin{pmatrix} \gamma_{1,\alpha} & 0 & \dots & 0 & \langle A_\alpha \nu, e_1 \rangle \\ 0 & \gamma_{2,\alpha} & \dots & 0 & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & \gamma_{n-1,\alpha} & \langle A_\alpha \nu, e_{n-1} \rangle \\ \langle A_\alpha \nu, e_1 \rangle & \dots & \dots & \langle A_\alpha \nu, e_{n-1} \rangle & \langle A_\alpha \nu, \nu \rangle \end{pmatrix}.$$

### 3. ALGEBRAIC FORMULAS

Let us prove the algebraic relation (2.8), to do so, we put

$$\rho_\alpha = \langle \eta, N_\alpha \rangle, \quad \mu_\alpha = \langle V, N_\alpha \rangle.$$

Moreover, we will write  $v \leq u$  for multi-indices  $v, u \in \mathbb{N}^q$  if the difference  $u - v \in \mathbb{N}^q$ . For a multi-index  $u = (u_1, \dots, u_q) \in \mathbb{N}^q$  of length  $|u| = n$ , we set

$$\binom{n}{u} = \frac{n!}{u!} = \frac{n!}{u_1! \dots u_q!}.$$

**Proposition 1.** *Let  $\tilde{A} = (A_1|_\Sigma, \dots, A_q|_\Sigma) = (\rho_1 A_\Sigma + \mu_1 I, \dots, \rho_q A_\Sigma + \mu_q I)$ .*

*Put  $\tilde{\sigma}_u = \sigma_u(\tilde{A})$ . Then for every multi-index  $u \in \mathbb{N}^q$  we have*

$$(3.1) \quad \tilde{\sigma}_u = \frac{1}{(n-1-|u|)!} \sum_{l \leq u} \binom{|l|}{l} \rho^l \mu^{u-l} \binom{n-1-|l|}{u-l} \sigma_{|l|}(A_\Sigma).$$

*Proof.* First, by applying the following relation (see [6])

$$\sigma_u(aA_1, \dots, A_q) = a^{u_1} \sigma_u(A_1, \dots, A_q)$$

we obtain

$$(3.2) \quad \tilde{\sigma}_u(\rho_1 A_\Sigma + \mu_1 I, \dots, \rho_q A_\Sigma + \mu_q I) = \rho^u \tilde{\sigma}_u(A_\Sigma + \theta_1 I, \dots, A_\Sigma + \theta_q I)$$

where

$$\theta_\alpha = \frac{\mu_\alpha}{\rho_\alpha} \text{ and } \rho^u = \rho_1^{u_1} \dots \rho_q^{u_q}.$$



Consider the GNT  $T_u = T_u(\hat{A})$ , with

$$\hat{A} = (A_\Sigma + \theta_1 I, \dots, A_\Sigma + \theta_q I).$$

The characteristic polynomial associate to  $\hat{A}$  is then

$$P_{\hat{A}}(t) = \sum_{|u| \leq n-1} \hat{\sigma}_u t^u$$

Moreover, we have

$$(3.3) \quad P_{\hat{A}}(t) = \det \left( I + \sum_{\alpha=1}^q t_\alpha (A_\Sigma + \theta_\alpha) I \right)$$

$$(3.4) \quad = \prod_{j=1}^{n-1} (1 + \tau_j (t_1 + \dots + t_q) + t_1 \theta_1 + \dots + t_q \theta_q).$$

By expanding the linear factorization of a monic polynomial we obtain

$$P_{\hat{A}}(t) = \sum_{j=0}^{n-1} (1 + t_1 \theta_1 + \dots + t_q \theta_q)^{n-1-j} (t_1 + \dots + t_q)^j \sigma_j(A_\Sigma)$$

and by the multinomial theorem we get

$$P_{\hat{A}}(t) = \sum_{j=0}^{n-1} \sum_{k=0}^{n-1-j} \sum_{|v|=k} \sum_{|l|=j} \frac{1}{(n-1-j-k)!} \binom{j}{l} \binom{n-1-j}{v} \sigma_j(A_\Sigma) \theta^v t^{v+l}.$$

where  $l = (l_1, \dots, l_q)$ ,  $v = (v_1, \dots, v_q)$ ,  $\theta = (\theta_1, \dots, \theta_q)$  and  $t = (t_1, \dots, t_q)$ . Let  $u = v + l$  so  $j = |u| - |v|$  and

$$P_{\hat{A}}(t) = \sum_{|u| \leq n-1} \sum_{l \leq u} \frac{1}{(n-1-|u|)!} \binom{|l|}{l} \binom{n-1-|l|}{u-l} \theta^{u-l} \sigma_{|l|}(A_\Sigma) t^u.$$

Thus the coefficient before  $t^u$  is given by

$$\hat{\sigma}_u(A_\Sigma + \theta_1 I, \dots, A_\Sigma + \theta_q I) = \frac{1}{(n-1-|u|)!} \sum_{l \leq u} \binom{|l|}{l} \theta^{u-l} \binom{n-1-|l|}{u-l} \sigma_{|l|}(A_\Sigma).$$

Consequently

$$(3.5) \quad \tilde{\sigma}_u(\rho_1 A_\Sigma + \mu_1 I, \dots, \rho_q A_\Sigma + \mu_q I) = \frac{1}{(n-1-|u|)!} \sum_{l \leq u} \binom{|l|}{l} \rho^l \mu^{u-l} \binom{n-1-|l|}{u-l} \sigma_{|l|}(A_\Sigma)$$

On the other hand

Denote by  $\bar{A} = (A_\Sigma, \mu_1 I, \dots, \mu_q I)$  and  $\bar{\sigma}_{(k,v)} = \sigma_{(k,v)}(\bar{A})$ . We have

$$\sigma_{(k,v)}(\bar{A}) = \mu^v \sigma_{(k,v)}(A_\Sigma, I, \dots, I).$$

Now, we use the following formula ( see [6] )

$$\sigma(v, k)(B_1, \dots, B_q, I) = \binom{n-1-|v|}{k} \sigma_v(B_1, \dots, B_q)$$

consecutively  $q$ -times and we get

$$\begin{aligned} \sigma_{(j,v)}(A_\Sigma, I, \dots, I) &= \binom{n-1-|v|+v_1}{v_1} \dots \binom{n-1-|v|+|v|}{v_q} \sigma_j(A_\Sigma) \\ &= \frac{1}{(n-1-|v|-j)!} \binom{n-1-j}{v} \sigma_j(A_\Sigma). \end{aligned}$$

Taking  $j = |u| - |v|$ , we obtain

$$= \frac{1}{(n-1-|u|)!} \binom{n-1-|u|+|v|}{v} \sigma_{|u|-|v|}(A_\Sigma)$$

and if we let  $l = u - v$ , we have

$$(3.6) \quad \sigma_{(|l|, u-l)}(A_\Sigma, I, \dots, I) = \frac{1}{(n-1-|u|)!} \binom{n-1-|l|}{u-l} \sigma_{|l|}(A_\Sigma)$$

So by relations (3.2) and (3.6), we get

$$(3.7) \quad \tilde{\sigma}_u = \frac{1}{(n-1-|u|)!} \sum_{l \leq u} \binom{|l|}{l} \rho^l \mu^{u-l} \binom{n-1-|l|}{u-l} \sigma_{|l|}(A_\Sigma)$$

□

Notice, that in the proof of the above Proposition, we assumed  $\rho_\alpha \neq 0$  for all  $\alpha$ , this allowed us to defined the constants  $\theta_\alpha$ . The assumption  $\rho_\alpha \neq 0$  is not necessary. Since if there exists  $\alpha \in \{1, \dots, q\}$  such that  $\rho_\alpha = 0$ . then (see [6])

$$\begin{aligned} \tilde{\sigma}_u &= \tilde{\sigma}_u(\rho_1 A_\Sigma + \mu_1 I, \dots, \rho_q A_\Sigma + \mu_q I) \\ &= \tilde{\sigma}_u(\rho_1 A_\Sigma + \mu_1 I, \dots, \rho_{i-1} A_\Sigma + \mu_{i-1} I, \mu_i I, \rho_{i+1} A_\Sigma + \mu_{i+1} I, \dots, \rho_q A_\Sigma + \mu_q I) \\ &= \mu_i^{u_i} \tilde{\sigma}_{\tilde{u}}(\rho_1 A_\Sigma + \mu_1 I, \dots, \rho_{i-1} A_\Sigma + \mu_{i-1} I, \rho_{i+1} A_\Sigma + \mu_{i+1} I, \dots, \rho_q A_\Sigma + \mu_q I), \end{aligned}$$

where

$$\tilde{u} = (u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_q)$$

and we may apply the above Proposition to  $\tilde{\sigma}_{\tilde{u}}$ .

**Proposition 2.** *Let  $\tilde{A} = (A_1|_\Sigma, \dots, A_q|_\Sigma)$ , where  $A_\alpha|_\Sigma = (\rho_\alpha A_\Sigma + \mu_\alpha I)$ , and  $\overline{A} = (A_\Sigma, \mu_1 I, \dots, \mu_q I)$ . For any multi-index  $u \in \mathbb{N}^q$ , put  $\tilde{\sigma}_u = \sigma_u(\tilde{A})$  and  $\overline{\sigma}_u = \sigma_u(\overline{A})$ . Then we have*

$$(3.8) \quad \tilde{\sigma}_u = \sum_{l \leq u} \binom{|l|}{l} \rho^l \overline{\sigma}_{(|l|, u-l)}.$$

*Proof.* First notice that

$$\bar{\sigma}_{(j,v)} = \sigma_{(j,v)}(\bar{A}) = \mu^v \sigma_{(j,v)}(A_\Sigma, I, \dots, I).$$

Using the following formula (see [6])

$$\sigma_{(v,j)}(B_1, \dots, B_q, I) = \binom{n-1-|v|}{j} \sigma_v(B_1, \dots, B_q)$$

consecutively  $q$ -times, we obtain

$$\begin{aligned} \sigma_{(j,v)}(A_\Sigma, I, \dots, I) &= \binom{n-1-|v|+v_1}{v_1} \dots \binom{n-1-|v|+|v|}{v_q} \sigma_j(A_\Sigma) \\ &= \frac{1}{(n-1-|v|-j)!} \binom{n-1-j}{v} \sigma_j(A_\Sigma) \end{aligned}$$

or by putting  $l = u - v$

$$\sigma_{(|l|, u-l)} = \frac{1}{(n-1-|u|)!} \binom{n-1-|l|}{u-l} \sigma_{|l|}(A_\Sigma)$$

so

$$\bar{\sigma}_{(|l|, u-l)} = \frac{\mu^{u-l}}{(n-1-|u|)!} \binom{n-1-|l|}{u-l} \sigma_{|l|}(A_\Sigma)$$

Thus by (3.1) we get (3.8).  $\square$

Let us now state the relation between symmetric functions corresponding to the families  $(A_\alpha)$  and  $(A_\alpha|_\Sigma)$ . Notice that there are of different sizes.

**Proposition 3.** *Let  $A = (A_1, \dots, A_q)$  and  $\tilde{A} = (A_1|_\Sigma, \dots, A_q|_\Sigma)$ . Put  $\sigma_u = \sigma_u(A)$  and  $\tilde{\sigma}_u = \sigma_u(\tilde{A})$ . Then*

$$\begin{aligned} \sigma_u &= \tilde{\sigma}_u + \sum_{\alpha} C_{\alpha} \tilde{\sigma}_{\alpha_b(v)} \\ &\quad + \sum_{\alpha \neq \beta} \sum_{\#(0) \leq w \leq u} \sum_{\alpha, \beta} (-1)^{|w|-|v|+1} \binom{|u|-|w|}{u-w} B_{\alpha}^{\top} \tilde{A}^{u-w} B_{\beta} \tilde{\sigma}_{\alpha_b \beta_b(w)} \end{aligned}$$

where  $B_{\alpha}^{\top} = (\langle A_{\alpha} v, e_1 \rangle, \dots, \langle A_{\alpha} v, e_{n-1} \rangle)$  and  $C_{\alpha} = \langle A_{\alpha} v, v \rangle$ .

*Proof.* First calculate the characteristic polynomial  $P_A(t)$  of  $A = (A_1, \dots, A_q)$ . By definition, we have

$$\begin{aligned} P_A(t) &= \det(I + \sum_{\alpha} t_{\alpha} A_{\alpha}) \\ &= \det \left( I_{n-1} + \sum_{\alpha} t_{\alpha} A_{\alpha}|_{\Sigma} \right) \det \left( C - B^{\top} \left( I_{n-1} + \sum_{\alpha} t_{\alpha} A_{\alpha}|_{\Sigma} \right)^{-1} B \right) \end{aligned}$$

where  $B^{\top}$  and  $C$  are respectively given by

$$B^{\top} = \left( \sum_{\alpha} t_{\alpha} \langle A_{\alpha} v, e_1 \rangle, \dots, \sum_{\alpha} t_{\alpha} \langle A_{\alpha} v, e_{n-1} \rangle \right) = \sum_{\alpha} t_{\alpha} B_{\alpha}^{\top}$$

and

$$C = 1 + \sum_{\alpha} t_{\alpha} \langle A_{\alpha} v, v \rangle.$$

To simplify the expressions, we put

$$M = \left( I_{n-1} + \sum_{\alpha} t_{\alpha} A_{\alpha} |_{\Sigma} \right).$$

and

$$f(t) = \det(C - B^{\top} M^{-1} B).$$

Moreover it is clear that

$$B^{\top} M^{-1} B = \sum_{\alpha, \beta} t_{\alpha} t_{\beta} B_{\alpha}^{\top} M^{-1} B_{\beta},$$

therefore we get

$$f(t) = \left( 1 + \sum_{\alpha} t_{\alpha} C_{\alpha} - \sum_{\alpha, \beta} t_{\alpha} t_{\beta} B_{\alpha}^{\top} M^{-1} B_{\beta} \right).$$

As it is easily seen that

$$P_A(t) = f(t) P_{\tilde{A}}(t).$$

where  $P_A(t)$  and  $P_{\tilde{A}}(t)$  are defined by

$$P_A(t) = \sum_{|u| \leq n} \sigma_u t^u$$

and

$$P_{\tilde{A}}(t) = \sum_{|u| \leq n-1} \tilde{\sigma}_u t^u.$$

Thus we get

$$\frac{\partial^u}{\partial t^u} P_A(t) \big|_{t=0} = u! \sigma_u, \quad \frac{\partial^u}{\partial t^u} P_{\tilde{A}}(t) \big|_{t=0} = u! \tilde{\sigma}_u.$$

Similarly we obtain

$$\frac{\partial^u}{\partial t^u} (P_{\tilde{A}}(t) \cdot f(t)) \big|_{t=0} = \sum_{v \leq u} \binom{u}{v} \left( \frac{\partial^v}{\partial t^v} P_{\tilde{A}}(t) \cdot \frac{\partial^{u-v}}{\partial t^{u-v}} f(t) \right) \big|_{t=0},$$

where

$$\binom{u}{v} = \frac{u!}{v!} = \frac{u_1! \dots u_1!}{v_1! \dots v_q!}.$$

Now we will compute  $\frac{\partial^{u-v}}{\partial t^{u-v}} f(t)$ . For  $v = u$ , it is checked

$$\frac{\partial^{u-v}}{\partial t^{u-v}} f(t) \big|_{t=0} = f(0, \dots, 0) = 1.$$

If  $v = \alpha_b(u)$ , we have

$$\frac{\partial^{u-v}}{\partial t^{u-v}} f(t) \big|_{t=0} = \frac{\partial}{\partial t_{\alpha}} f(t) \big|_{t=0} = C_{\alpha}$$

and for any  $v \leq \alpha_b \beta_b(u)$ , we obtain

$$\frac{\partial^{u-v}}{\partial t^{u-v}} f(t) \big|_{t=0} = \frac{\partial^{u-v}}{\partial t^{u-v}} \left( 1 + \sum_{\alpha} t_{\alpha} C_{\alpha} - \sum_{\alpha, \beta} t_{\alpha} t_{\beta} B_{\alpha}^{\top} \left( I_{n-1} + \sum_{\alpha} t_{\alpha} A_{\alpha} |_{\Sigma} \right)^{-1} B_{\beta} \right) \big|_{t=0}.$$

Taking  $|t| < \varepsilon$  for some small enough  $\varepsilon$  we get

$$\left\| \sum_{\alpha} t_{\alpha} A_{\alpha} |_{\Sigma} \right\| \leq 1.$$

Thus

$$\left( I_{n-1} + \sum_{\alpha} t_{\alpha} A_{\alpha} |_{\Sigma} \right)^{-1} = \sum_{k=0}^{\infty} (-1)^k \left( \sum_{\alpha} t_{\alpha} A_{\alpha} |_{\Sigma} \right)^k$$

Since the matrices  $A_{\alpha} |_{\Sigma}$  are diagonal, then they commute. Therefore, we may write

$$\left( \sum_{\alpha} t_{\alpha} A_{\alpha} |_{\Sigma} \right)^k = \sum_{w_1 + \dots + w_q = k} \binom{k}{w} t^w \tilde{A}^w.$$

Hence

$$\left( I_{n-1} + \sum_{\alpha} t_{\alpha} A_{\alpha} |_{\Sigma} \right)^{-1} = \sum_{k=0}^{\infty} (-1)^k \left( \sum_{w_1 + \dots + w_q = k} \binom{k}{w} t^w \tilde{A}^w \right)$$

Which gives

$$\begin{aligned} \frac{\partial^{u-v}}{\partial t^{u-v}} f(t) \big|_{t=0} &= \frac{\partial^{u-v}}{\partial t^{u-v}} \left( - \sum_{\alpha, \beta} t_{\alpha} t_{\beta} B_{\alpha}^{\top} \sum_{k=0}^{\infty} (-1)^k \left( \sum_{w_1 + \dots + w_q = |w|=k} \binom{|w|}{w} t^w \tilde{A}^w \right) B_{\beta} \right) \big|_{t=0} \\ &= \frac{\partial^{u-v}}{\partial t^{u-v}} \left( - \sum_{\alpha, \beta} B_{\alpha}^{\top} \sum_{k=0}^{\infty} (-1)^k \left( \sum_{w_1 + \dots + w_q = |w|=k} \binom{|w|}{w} t^{w + \alpha^{\#} \beta^{\#}(0)} \tilde{A}^w \right) B_{\beta} \right) \big|_{t=0} \\ &= - \sum_{\alpha, \beta} \sum_{v \leq \alpha_b \beta_b(u)} B_{\alpha}^{\top} (-1)^{|u|-|v|-2} (u-v)! \binom{|u|-|v|-2}{u - \alpha^{\#} \beta^{\#}(v)} \tilde{A}^{u - \alpha^{\#} \beta^{\#}(v)} B_{\beta} \\ &= \sum_{\alpha, \beta} \sum_{v \leq \alpha_b \beta_b(u)} B_{\alpha}^{\top} (-1)^{|u|-|v|-1} (u-v)! \binom{|u|-|v|-2}{u - \alpha^{\#} \beta^{\#}(v)} \tilde{A}^{u - \alpha^{\#} \beta^{\#}(v)} B_{\beta}. \end{aligned}$$

Finally we obtain

$$\begin{aligned} u! \sigma_u &= u! \tilde{\sigma}_u + \sum_{\alpha} (\alpha_b(u))! \binom{u}{\alpha_b(u)} C_{\alpha} \tilde{\sigma}_{\alpha_b(u)} \\ &\quad + \sum_{\alpha, \beta} \sum_{v \leq u} \binom{u}{v} (-1)^{|u|-|v|-1} (u-v)! v! \binom{|u|-|v|-2}{u - \alpha_b \beta_b(v)} B_{\alpha}^{\top} \tilde{A}^{u - \alpha_b \beta_b(v)} B_{\beta} \tilde{\sigma}_v \end{aligned}$$

or equivalently,

$$\sigma_u = \tilde{\sigma}_u + \sum_{\alpha} C_{\alpha} \tilde{\sigma}_{\alpha_b(u)} + \sum_{\alpha, \beta} \sum_{0 \leq v \leq \alpha_b, \beta_b(u)} (-1)^{|u|-|v|-1} \binom{|u|-|v|-2}{u-\alpha^{\#}\beta^{\#}(v)} B_{\alpha}^{\top} \tilde{A}^{u-\alpha^{\#}\beta^{\#}(v)} B_{\beta} \tilde{\sigma}_v.$$

Hence

$$\sigma_u = \tilde{\sigma}_u + \sum_{\alpha} C_{\alpha} \tilde{\sigma}_{\alpha_b(u)} + \sum_{\alpha^{\#}\beta^{\#}(0) \leq w \leq u} \sum_{\alpha, \beta} (-1)^{|w|-|v|+1} \binom{|u|-|w|}{u-w} B_{\alpha}^{\top} \tilde{A}^{u-w} B_{\beta} \tilde{\sigma}_{\alpha_b, \beta_b(w)}.$$

□

#### 4. GENERALIZED NEWTON TRANSFORMATION ON THE BOUNDARY

We use the same notations as in previous sections. In this section we give the expression of the GNT  $T_u = T_u(\tilde{A})$ , where  $\tilde{A} = (A_1|_{\Sigma}, \dots, A_q|_{\Sigma})$ , on the boundary  $\Sigma^{n-1}$  of  $M^n$ . Recall that

$$A_{\alpha}|_{\Sigma} = \rho_{\alpha} A_{\Sigma} + \mu_{\alpha} I,$$

where

$$\rho_{\alpha} = \langle \eta, N_{\alpha} \rangle, \quad \mu_{\alpha} = \langle V, N_{\alpha} \rangle, \quad V = \sum_{\alpha=1}^q \lambda_{\alpha} \xi_{\alpha}.$$

**Proposition 4.** *Let  $\overline{M}^{n+q}$  be an  $(n+q)$ -Riemannian manifold and  $P^n \subset \overline{M}^{n+q}$  an oriented totally umbilical  $n$ -submanifold of  $\overline{M}^{n+q}$ . Denote by  $\Sigma^{n-1} \subset P^n$  an  $(n-1)$ -compact hypersurface of  $P^n$ . Let  $\Psi : M^n \rightarrow \overline{M}^{n+q}$  be an oriented connected and compact submanifold of  $\overline{M}^{n+q}$  with boundary  $\Sigma^{n-1} = \Psi(\partial M)$ . Then along the boundary  $\partial M$ , we have*

$$(4.1) \quad \langle T_u \nu, \nu \rangle = \tilde{\sigma}_u(A_1|_{\Sigma}, \dots, A_q|_{\Sigma}).$$

*Proof.* We make a recursive proof. Assume that (4.1) holds for any multi-index  $v < u$ . We have by the recurrence definition of  $T_u$

$$\begin{aligned} \langle T_u \nu, \nu \rangle &= \sigma_u \langle \nu, \nu \rangle - \sum_{\alpha} \langle A_{\alpha} T_{\alpha_b(u)} \nu, \nu \rangle \\ &= \sigma_u - \sum_{\alpha} \langle T_{\alpha_b(u)} \nu, A_{\alpha} \nu \rangle \\ &= \sigma_u - \sum_{\alpha} \langle T_{\alpha_b(u)} \nu, \nu \rangle \langle A_{\alpha} \nu, \nu \rangle - \sum_{\alpha, i} \langle T_{\alpha_b(u)} \nu, e_i \rangle \langle A_{\alpha} e_i, \nu \rangle. \end{aligned}$$

Put

$$C_{\alpha} = \langle A_{\alpha} \nu, \nu \rangle, \quad b_{i, \alpha} = \langle A_{\alpha} e_i, \nu \rangle$$

then

$$\langle T_u \nu, \nu \rangle = \sigma_u - \sum_{\alpha} C_{\alpha} \tilde{\sigma}_{\alpha_b(u)} - \sum_{\alpha, i} b_{i, \alpha} \langle T_{\alpha_b(u)} \nu, e_i \rangle.$$

Let us compute  $\langle T_u \nu, e_i \rangle$  for any multi-index  $u \in \mathbb{N}^q$ . Notice that

$$A_{\alpha} e_i = \rho_{\alpha} A_{\Sigma} e_i + \mu_{\alpha} e_i + b_{i, \alpha} \nu$$

assuming that  $\{e_1, \dots, e_{n-1}\}$  is an orthonormal basis of  $T_p \Sigma^{n-1}$  consisting of eigenvectors of  $A_\Sigma$  (with eigenvalues  $\tau_i$ ). Thus

$$A_\alpha e_i = \gamma_{i,\alpha} e_i + b_{i,\alpha} \nu$$

where

$$\gamma_{i,\alpha} = \rho_\alpha \tau_i + \mu_\alpha.$$

We will show inductively that

$$(4.2) \quad \langle T_u \nu, e_i \rangle = \sum_{\alpha} \sum_{\alpha^\#(0) \leq w \leq u} (-1)^{|u|-|w|+1} \binom{|u|-|w|}{u-w} b_{i,\alpha} \gamma_i^{u-w} \tilde{\sigma}_{\alpha_b(w)}$$

where

$$\gamma_i = (\gamma_{i,1}, \dots, \gamma_{i,q})$$

Indeed, for  $u = \beta^\#(0)$  we have,

$$\begin{aligned} \langle T_u \nu, e_i \rangle &= \sigma_{\beta^\#(0)} \langle \nu, e_i \rangle - \sum_{\alpha} \langle A_\alpha T_{\alpha_b(\beta^\#(0))} \nu, \nu \rangle \\ &= - \sum_{\alpha} \langle A_\alpha \nu, \nu \rangle \\ &= -b_{i,\beta} \\ &= \sum_{\alpha} \sum_{\alpha^\#(0) \leq w \leq \beta^\#(0)} (-1)^{|\beta^\#(0)|-|w|+1} \binom{|\beta^\#(0)|-|w|}{\beta^\#(0)-w} b_{i,\alpha} \gamma_i^{\beta^\#(0)-w} \tilde{\sigma}_{\alpha_b(w)}, \end{aligned}$$

since the sum reduces to one element for  $\alpha = \beta$ .

Assume that (4.2) holds for all multi index  $v < u$ . Then again by the recursive definition of  $T_u$  we get

$$\begin{aligned} \langle T_u \nu, e_i \rangle &= \sigma_u \langle \nu, e_i \rangle - \sum_{\alpha} \langle A_\alpha T_{\alpha_b(u)} \nu, e_i \rangle \\ &= - \sum_{\alpha} \langle T_{\alpha_b(u)} \nu, A_\alpha e_i \rangle \\ &= - \sum_{\alpha} \langle T_{\alpha_b(u)} \nu, \nu \rangle \langle A_\alpha e_i, \nu \rangle - \sum_{\alpha, j} \langle T_{\alpha_b(u)} \nu, e_i \rangle \langle A_\alpha e_i, e_j \rangle \\ &= - \sum_{\alpha} \langle T_{\alpha_b(u)} \nu, e_i \rangle \gamma_{i,\alpha} - \sum_{\alpha} b_{i,\alpha} \tilde{\sigma}_{\alpha_b(u)} \\ &= - \sum_{\alpha} \gamma_{i,\alpha} \left( \sum_{\beta} \sum_{\beta^\#(0) \leq w \leq \alpha_b(u)} (-1)^{|u|-|w|} \binom{|u|-|w|-1}{\alpha_b(u)-w} b_{i,\beta} \gamma_i^{\alpha_b(u)-w} \tilde{\sigma}_{\beta_b(w)} \right) \\ &\quad - \sum_{\alpha} b_{i,\alpha} \tilde{\sigma}_{\alpha_b(u)}. \end{aligned}$$

Clearly

$$\gamma_{i,\alpha} \gamma_i^{\alpha_b(u)-w} = \gamma_i^{u-w}$$

Notice that taking  $w = u$  we get the last sum. Thus we obtain

$$\langle T_u \nu, e_i \rangle = \sum_{\alpha, \beta} \left( \sum_{\beta^\#(0) \leq w \leq u} (-1)^{|u|-|w|+1} \binom{|u|-|w|-1}{\alpha_b(u)-w} b_{i,\beta} \gamma_i^{u-w} \tilde{\sigma}_{\beta_b(w)} \right).$$

Since

$$\sum_{\alpha} \binom{|u|-|w|-1}{\alpha_b(u)-w} = \binom{|u|-|w|}{u-w}$$

we deduce

$$\langle T_u \nu, e_i \rangle = \sum_{\beta} \left( \sum_{\beta^\#(0) \leq w \leq u} (-1)^{|u|-|w|+1} \binom{|u|-|w|}{u-w} b_{i,\beta} \gamma_i^{u-w} \tilde{\sigma}_{\beta_b(w)} \right)$$

which end the proof of (4.2).

Now, we may prove (4.1). First, we have

$$\sum_{\alpha, i} \langle T_{\alpha_b(u)} \nu, e_i \rangle b_{i,\alpha} = \sum_{\alpha, \beta, i} \left( \sum_{\beta^\#(0) \leq w \leq \alpha_b(u)} (-1)^{|u|-|w|} \binom{|u|-|w|-1}{\alpha_b(u)-w} b_{i,\beta} \gamma_i^{\alpha_b(u)-w} \tilde{\sigma}_{\beta_b(w)} \right)$$

Replacing  $w$  by  $\alpha_b \beta_b(w)$ , we obtain

$$\sum_{\alpha, i} \langle T_{\alpha_b(u)} \nu, e_i \rangle b_{i,\alpha} = \sum_{\alpha, \beta, i} \left( \sum_{\alpha^\# \beta^\#(0) \leq w \leq u} (-1)^{|u|-|w|+1} \binom{|u|-|w|}{u-w} b_{i,\alpha} \gamma_i^{u-w} b_{i,\beta} \tilde{\sigma}_{\alpha_b \beta_b(w)} \right)$$

Noticing that

$$\sum_i b_{i,\alpha} \gamma_i^{u-w} b_{i,\beta} = B_\alpha^\top \tilde{A}^{u-w} B_\beta$$

we infer

$$\sum_{\alpha, i} \langle T_{\alpha_b(u)} \nu, e_i \rangle b_{i,\alpha} = \sum_{\alpha, \beta} \sum_{\alpha^\# \beta^\#(0) \leq w \leq u} (-1)^{|u|-|w|+1} \binom{|u|-|w|}{u-w} B_\alpha^\top \tilde{A}^{u-w} B_\beta \cdot \tilde{\sigma}_{\alpha_b \beta_b(w)}$$

Summing up all the above considerations we get

$$\begin{aligned} \langle T_u \nu, \nu \rangle &= \sigma_u - \sum_{\alpha} \langle T_{\alpha_b(u)} \nu, \nu \rangle \langle A_\alpha \nu, \nu \rangle - \sum_{\alpha, i} \langle T_{\alpha_b(u)} \nu, e_i \rangle \langle A_\alpha e_i, \nu \rangle \\ &= \sigma_u - \sum_{\alpha} C_\alpha \tilde{\sigma}_{\alpha_b(u)} - \sum_{\alpha, \beta} \sum_{\alpha^\# \beta^\#(0) \leq w \leq u} (-1)^{|u|-|w|+1} \binom{|u|-|w|}{u-w} B_\alpha^\top \tilde{A}^{u-w} B_\beta \cdot \tilde{\sigma}_{\alpha_b \beta_b(w)} \end{aligned}$$

or, equivalently,

$$\sigma_u = \tilde{\sigma}_u + \sum_{\alpha} C_\alpha \tilde{\sigma}_{\alpha_b(u)} + \sum_{\alpha, \beta} \sum_{\alpha^\# \beta^\#(0) \leq w \leq u} (-1)^{|u|-|w|+1} \binom{|u|-|w|}{u-w} B_\alpha^\top \tilde{A}^{u-w} B_\beta \cdot \tilde{\sigma}_{\alpha_b \beta_b(w)}$$

Applying Proposition (3), we obtain

$$\langle T_u \nu, \nu \rangle = \tilde{\sigma}_u.$$

□



By Proposition (1) we obtain the following expression of  $\langle T_u \nu, \nu \rangle$  in terms of symmetric functions of the shape operator  $A_\Sigma$ .

**Corollary 1.** *With the conditions of Proposition (4), we have*

$$(4.3) \quad \langle T_u \nu, \nu \rangle = \frac{1}{n-1-|u|} \sum_{l \leq |u|} \binom{n-1-|l|}{|u|-l} \rho^l \mu^{u-l} \sigma_{|l|}(A_\Sigma).$$

The relation (4.3) will be more simple if we suppose that the embedding  $P^n \subset \overline{M}^{n+q}$  is totally geodesic.

**Corollary 2.** *With the conditions of Proposition (4) and assuming that  $P^n \subset \overline{M}^{n+q}$  is totally geodesic, then for every multi-index  $u$  with length  $|u| \leq n-1$ , we have*

$$\langle T_u \nu, \nu \rangle = \rho^u \sigma_{|u|}(A_\Sigma)$$

*Proof.* It suffices to use (4.3) with  $\mu_\alpha = 0$ . □

## 5. TRANSVERSALITY OF SUBMANIFOLDS

The formula for the generalized Newton transformation implies the relation between transversality of  $M^n$  and  $P^n$  and ellipticity of  $T_u$  provided that  $P^n$  is totally geodesic in  $\overline{M}^{n+1}$ . This generalizes the result in ([2]) to any arbitrary codimension.

**Theorem 1.** *With the conditions in Corollary (2) the submanifolds  $M^n$  and  $P^n$  are transversal along  $\partial M$  provided that for some multi-index  $u$  of length  $1 \leq |u| \leq n-1$ , the generalized Newton transformation  $T_u$  is positive definite on  $M^n$ .*

*Proof.* Saying that  $M^n$  and  $P^n$  are not transversal means that there exist  $p \in \partial M$  such that for every  $\alpha \in \{1, \dots, q\}$  we have

$$\rho_u = \langle \eta, N_\alpha \rangle = 0 \quad \text{at } p.$$

Therefore, if we suppose that for all  $p \in \overline{M}^{n+q}$ ,  $T_u$  is positive definite, then by Corollary 7,  $\rho^u(p) \neq 0$ . Thus

$$\langle \eta, N_\alpha \rangle \neq 0,$$

hence  $M^n$  and  $P^n$  are transversal. □

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